

DETERMINING A BOUNDARY COEFFICIENT IN A DISSIPATIVE WAVE EQUATION: UNIQUENESS AND DIRECTIONAL LIPSCHITZ STABILITY

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ABSTRACT. We are concerned with the problem of determining the damping boundary coefficient appearing in a dissipative wave equation from a single boundary measurement. We prove that the uniqueness holds at the origin provided that the initial condition is appropriately chosen. We show that the choice of the initial condition leading to uniqueness is related to a fine version of unique continuation property for elliptic operators. We also establish a Lipschitz directional stability estimate at the origin, which is obtained by a linearization process.

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1. INTRODUCTION

Let Ω be a C^∞ -smooth bounded domain of \mathbb{R}^d with boundary Γ . We assume that Γ can be partitioned into two disjoint closed subsets with non empty interior that are denoted by Γ_0 and Γ_1 .

We set, where $\tau > 0$ is fixed,

$$Q = \Omega \times (0, \tau), \quad \Sigma_0 = \Gamma_0 \times (0, \tau), \quad \Sigma_1 = \Gamma_1 \times (0, \tau),$$

and we consider the following initial-boundary value problem (abbreviated to IBVP in the sequel) for the wave equation:

$$(1) \quad \begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma_0, \\ \partial_\nu u + b(x)\partial_t u = 0 & \text{on } \Sigma_1, \\ u(\cdot, 0) = u^0, \quad \partial_t u(\cdot, 0) = v^0. \end{cases}$$

We are interested in the inverse problem of determining the boundary coefficient b from the boundary measurement $\partial_\nu u_b|_{\Sigma_1}$, where u_b is the solution, if it exists, of the IBVP (1). At least formally, if each term in the left hand side of the third

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equation of (1) belongs to $L^2(\Sigma_1)$, then the inverse problem under consideration is equivalent to determining b from $b\partial_t u_b|_{\Sigma_1}$. Therefore this problem is highly non linear.

Before stating our uniqueness and stability results, we need to reformulate the IBVP (1) as an abstract Cauchy problem. To this purpose, we set

$$V = \{w \in H^1(\Omega); w|_{\Gamma_0} = 0\}.$$

Here $w|_{\Gamma_0}$ is to be understood in the usual trace sense. When it is equipped with the H^1 -norm, V is a Hilbert space. We note in addition that the Poincaré inequality holds true for V and therefore $w \mapsto \|\nabla w\|_{L^2(\Omega)^d}$ defines an equivalent norm on V .

For $s > 0$ and $1 \leq r \leq \infty$, we introduce the vector space

$$B_{s,r}(\mathbb{R}^{d-1}) := \{w \in \mathcal{S}'(\mathbb{R}^{d-1}); (1 + |\xi|^2)^{s/2} \widehat{w} \in L^r(\mathbb{R}^{d-1})\},$$

where $\mathcal{S}'(\mathbb{R}^{d-1})$ is the space of temperate distributions on \mathbb{R}^{d-1} and \widehat{w} is the Fourier transform of w . Equipped with its natural norm

$$\|w\|_{B_{s,r}(\mathbb{R}^{d-1})} := \|(1 + |\xi|^2)^{s/2} \widehat{w}\|_{L^r(\mathbb{R}^{d-1})},$$

$B_{s,r}(\mathbb{R}^{d-1})$ is a Banach space (it is noted that $B_{s,2}(\mathbb{R}^{d-1})$ is merely the usual Sobolev space $H^s(\mathbb{R}^{d-1})$). By using local charts and a partition of unity, we construct $B_{s,r}(\Gamma_1)$ from $B_{s,r}(\mathbb{R}^{d-1})$ similarly as $H^s(\Gamma_1)$ is built from $H^s(\mathbb{R}^{d-1})$.

The main interest in this spaces is that the multiplication by a function from $B_{s,1}(\Gamma_1)$ defines a bounded operator on $H^s(\Gamma_1)$ (we refer to [3][Theorem 2.1, page 605] for more details). We set

$$B_{1/2,1}^+(\Gamma_1) = \{b \in B_{1/2,1}(\Gamma_1); 0 \leq b\}.$$

Let $b \in B_{1/2,1}^+(\Gamma_1)$. We define on $\mathcal{H} = V \times L^2(\Omega)$ the unbounded operator \mathcal{A}_b as follows

$$\mathcal{A}_b \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

with domain

$$D(\mathcal{A}_b) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in [H^2(\Omega) \cap V] \times V; \partial_\nu u + bv = 0 \text{ on } \Gamma_1 \right\}.$$

From [8][Proposition 3.9.2, page 109], \mathcal{A}_b is an m-dissipative operator and therefore it generates on \mathcal{H} a C_0 -semigroup of contractions $e^{t\mathcal{A}_b}$. In that case, we have, for any integer $k \geq 1$,

$$(2) \quad e^{t\mathcal{A}_b} \begin{pmatrix} u^0 \\ v^0 \end{pmatrix} \in \bigcap_{j=0}^k C^{k-j}([0, \tau]; D(\mathcal{A}_b^j)), \quad \begin{pmatrix} u^0 \\ v^0 \end{pmatrix} \in D(\mathcal{A}_b^k),$$

with the convention that $\mathcal{A}_b^0 = I$ and $D(\mathcal{A}_b^0) = \mathcal{H}$.

Moreover, we have the estimate

$$(3) \quad \left\| e^{t\mathcal{A}_b} \begin{pmatrix} u^0 \\ v^0 \end{pmatrix} \right\|_{\bigcap_{j=0}^k C^{k-j}([0, \tau]; D(\mathcal{A}_b^j))} \leq C \left\| \begin{pmatrix} u^0 \\ v^0 \end{pmatrix} \right\|_{D(\mathcal{A}_b^k)}.$$

Here the constant C doesn't depend on u^0 and v^0 .

Let $A : D(A) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ be the unbounded operator given by

$$A = \Delta, \quad D(A) = \{w \in H^2(\Omega) \cap V; \partial_\nu w|_{\Gamma_1} = 0\}.$$

The following observation will be useful in the sequel: for any integer $k \geq 1$ and $b \in B_{1/2,1}^+(\Gamma_1)$, we have

$$(4) \quad D(A^k) \times \{0\} \subset D(\mathcal{A}_b^k).$$

We are now ready to state our main results.

Theorem 1.1. *Let $b \in B_{1/2,1}^+(\Gamma_1)$.*

- (a) $\mathcal{J} = \left\{ \begin{pmatrix} u^0 \\ 0 \end{pmatrix}; u^0 \in D(A) \text{ and } \partial_t u_0 \neq 0 \text{ a.e. on } \Sigma_1 \right\} \neq \emptyset.$
 (b) *Let $\begin{pmatrix} u^0 \\ 0 \end{pmatrix} \in \mathcal{J}$. Then $\partial_\nu u_b|_{\Sigma_1} = \partial_\nu u_0|_{\Sigma_1}$ implies $b = 0$.*

Theorem 1.2. *We fix $b \in B_{1/2,1}^+(\Gamma_1)$ non identically equal to zero. We assume*

$$(5) \quad \begin{pmatrix} u^0 \\ v^0 \end{pmatrix} \in \bigcap_{0 \leq \rho \leq 1} D(\mathcal{A}_{\rho b}^7)$$

and

$$(6) \quad b \partial_t u_0 \not\equiv 0.$$

Then there exists $0 < \rho_0 \leq 1$ so that

$$\tilde{\kappa} \|\rho b - 0\|_{B_{1/2,1}(\Gamma_1)} \leq \|\partial_\nu u_{\rho b} - \partial_\nu u_0\|_{L^2(\Sigma_1)}, \quad 0 \leq \rho \leq \rho_0.$$

Here $\tilde{\kappa}$ is a constant independent on ρ .

Remark 1. From the proof of Theorem 1.1 (a), we deduce

$$\mathcal{J}^\infty = \left\{ \begin{pmatrix} u^0 \\ 0 \end{pmatrix}; u^0 \in C^\infty(\overline{\Omega}) \text{ and } \partial_t u_0 \neq 0 \text{ a.e. on } \Sigma_1 \right\} \neq \emptyset.$$

Therefore, we can replace in Theorem 1.2 (5) and (6) by

$$\begin{pmatrix} u^0 \\ v^0 \end{pmatrix} = \begin{pmatrix} u^0 \\ 0 \end{pmatrix} \in \mathcal{J}^\infty.$$

The authors have already obtained in [1] a log-type stability estimate for the inverse problem consisting in determining both the potential and the damping coefficient in a dissipative wave equation from boundary measurements. These measurements correspond to all possible choices of the initial condition. The proofs in [1] are essentially based on observability inequalities for exactly controllable systems and spectral decompositions.

The problem of determining a potential in a wave equation from the so-called Dirichlet-to-Neumann was map studied by many authors. This problem was initiated by Rakesh and Symes [6]. We refer the reader who want to learn more on this problem to [2] and references therein.

The rest of this text is devoted to the proof of our main results. We prove Theorem 1.1 in Section 2 and Theorem 1.2 in Section 3.

2. PROOF OF THEOREM 1.1

We first prove a preliminary result. Henceforth, \mathcal{L}^k denotes the k -dimensional Lebesgue measure.

Lemma 2.1. *Let $\lambda \in \mathbb{R}$ and $u \in H^2(\Omega) \cap C^1(\overline{\Omega})$ satisfying*

$$\Delta u + \lambda u = 0 \text{ in } \Omega \quad \text{and} \quad \partial_\nu u = 0 \text{ on } \Gamma_1.$$

Then

$$\mathcal{L}^{d-1}(\{x \in \Gamma_1; u(x) = 0\}) = 0.$$

Sketch of the proof. Since Ω is C^∞ -smooth. Γ_i can be covered by a finite number of open subsets U , where U is such that there exists a C^∞ -diffeomorphism $\psi : U \rightarrow B$, $B = B(0, 1)$, satisfying

$$\psi(U \cap \Omega) \subset B_+, \quad \psi(U \cap \Gamma) \subset B_0,$$

with

$$B_+ = \{x = (x', x_d) \in B; x_d > 0\}, \quad B_0 = \{x = (x', x_d) \in B; x_d = 0\}.$$

We set $v(y) = u(\psi^{-1}(y))$, $y \in B_+$. Then $Pv = 0$ in B_+ and $\partial_d v = 0$ on B_0 . Here P is a second order operator with C^∞ coefficients. We extend v to the whole of B by setting $w = v$ in B_+ and $w(x', -x_d) = v(x', x_d)$, $(x', x_d) \in B_+$. We have $w \in H^2(\Omega) \cap C^1(\overline{B})$ because $\partial_d v = 0$ on B_0 and $Qw = 0$ in B , where Q is a second order operator whose coefficients are obtained by taking the even extension of the coefficients of P . Checking the details of this construction, we see that Q has Lipschitz coefficients.

Let $\epsilon > 0$ be given. If we denote by \mathcal{H}^k the k -dimensional Hausdorff measure, we get by applying [7][Theorem 2, page 342] that

$$\mathcal{H}^{d-2+\epsilon}(\{z \in B; w(z) = 0, \nabla w(z) = 0\}) = 0.$$

Therefore,

$$\mathcal{H}^{d-2+\epsilon}(\{y \in B_0; v(y) = 0, \nabla v(y) = 0\}) = 0,$$

In particular

$$\mathcal{L}^{d-1}(\{y \in B_0; v(y) = 0, \nabla v(y) = 0\}) = 0.$$

On the other hand by [4][Lemma 7.7, page 152], $\nabla_{y'} v(\cdot, 0) = 0$ a.e. in any set where $v(\cdot, 0)$ is constant. Hence

$$\mathcal{L}^{d-1}(\{y \in B_0; v(y) = 0, \partial_d v(y) = 0\}) = 0.$$

Bearing in mind that a Lipschitz map preserves Lebesgue sets of zero measure, we get

$$\mathcal{L}^{d-1}(\{x \in \Gamma_1; u(x) = 0, \partial_\nu u(x) = 0\}) = 0.$$

But, $\partial_\nu u = 0$ on Γ_1 . Hence

$$\mathcal{L}^{d-1}(\{x \in \Gamma_1; u(x) = 0\}) = 0.$$

□

The following regularity theorem will be useful in the sequel. Since this result is not explicitly recorded in the literature, for sake of completeness we sketch its proof.

Theorem 2.2. *Let $m \geq 0$ be an integer. For any $f \in H^m(\Omega)$, $g \in H^{m+3/2}(\Gamma_0)$ and $h \in H^{m+1/2}(\Gamma_1)$, the boundary value problem*

$$\begin{cases} -\Delta u = f & \text{in } Q, \\ u = g & \text{in } \Gamma_0, \\ \partial_\nu u = h & \text{in } \Gamma_1. \end{cases}$$

has a unique $u \in H^{m+2}(\Omega)$ satisfying

$$(7) \quad \|u\|_{H^{2+m}(\Omega)} \leq C (\|f\|_{H^m(\Omega)} + \|g\|_{H^{m+3/2}(\Gamma_0)} + \|h\|_{H^{m+1/2}(\Gamma_1)}).$$

Here the constant C is independent on f , g and h .

Proof. Since there exists $E \in H^{m+2}(\Omega)$ such that $E = g$ on Γ_0 and $\partial_\nu E = h$ in Γ with

$$\|E\|_{H^{2+m}(\Omega)} \leq C (\|g\|_{H^{m+3/2}(\Gamma_0)} + \|h\|_{H^{m+1/2}(\Gamma_1)}).$$

(e.g. for instance [5][Theorem 8.3, page 39]), we see, replacing u by $u - E$ and f by $f - \Delta E$, that it is enough to prove the theorem with $(g, h) = (0, 0)$. We consider then the BVP

$$(8) \quad \begin{cases} -\Delta u = f & \text{in } Q, \\ u = 0 & \text{in } \Gamma_0, \\ \partial_\nu u = 0 & \text{in } \Gamma_1. \end{cases}$$

Let $f \in L^2(\Omega)$. Because $w \in V \rightarrow \|\nabla w\|_{L^2(\Omega)^d}$ defines an equivalent norm on V , by the Lax-Milgram lemma, there exists a unique variational solution $u \in H^1(\Omega)$ of the BVP (8). That is

$$\int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega} f v dx \quad \text{for all } v \in V.$$

Let (Ω_0, Ω_1) be an open covering of an open neighborhood of $\overline{\Omega}$ such that $\Gamma_i \subset \Omega_i$ and $\Gamma_i \cap \Omega_{1-i} = \emptyset$, $i = 0, 1$. Let (ψ_0, ψ_1) be a partition of unity subordinate to the covering (Ω_0, Ω_1) with $\psi_i \in C_0^\infty(\Omega_i)$ and $\psi_i = 1$ in a neighborhood of Γ_i , $i = 0, 1$.

Let $u_i = u\psi_i$, $i = 0, 1$. Then a straightforward computation shows that u_0 and u_1 are the respective solutions of the variational problems

$$\begin{aligned} \int_{\Omega} \nabla u_0 \cdot \nabla v dx &= \int_{\Omega} f_0 v dx \quad \text{for all } v \in H_0^1(\Omega), \\ \int_{\Omega} \nabla u_1 \cdot \nabla v dx &= \int_{\Omega} f_1 v dx \quad \text{for all } v \in H^1(\Omega). \end{aligned}$$

Here

$$f_i = -2\nabla \psi_i \cdot \nabla u + \psi_i f, \quad i = 0, 1.$$

Since the regularity theorem [5][Theorem 5.4, page 165] is valid for both the Dirichlet and the Neumann BVP's, we obtain that $u_i \in H^2(\Omega)$ and

$$\|u_i\|_{H^2(\Omega)} \leq C \|f_i\|_{L^2(\Omega)} \leq C' (\|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}) \leq C'' \|f\|_{L^2(\Omega)}, \quad i = 0, 1.$$

Therefore $u = u_0 + u_1 \in H^2(\Omega)$ and

$$\|u\|_{H^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}.$$

Next, if $f \in H^1(\Omega)$ then $f_i \in H^1(\Omega)$, $i = 0, 1$. We can then repeat the previous argument to conclude that $u \in H^3(\Omega)$ and estimate (7) holds with $m = 1$. We complete the proof by using an induction argument in m . \square

Proof of Theorem 1.1. (a) Let (λ_k) be the sequence of eigenvalues, counted according to their multiplicity, of the unbounded operator $-A$. Let (φ_k) be a sequence of eigenfunctions forming an orthonormal basis of $L^2(\Omega)$, each φ_k corresponds to λ_k .

We note that, according to Theorem 2.2,

$$\varphi_k \in \bigcap_{m \in \mathbb{N}} H^m(\Omega) = C^\infty(\overline{\Omega}).$$

We fix k and we take $\begin{pmatrix} u^0 \\ v^0 \end{pmatrix} = \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix}$. Then $\begin{pmatrix} u^0 \\ v^0 \end{pmatrix} \in D(\mathcal{A}_0) \cap D(\mathcal{A}_b)$ and $u_0 = \cos(\sqrt{\lambda_k}t)\varphi_k$ is the solution of the IBVP (1) corresponding to this particular choice of $\begin{pmatrix} u^0 \\ v^0 \end{pmatrix}$.

We have $\partial_t u_0 = -\sqrt{\lambda_k} \sin(\sqrt{\lambda_k}t)\varphi_k$. Hence $\partial_t u_0 \neq 0$ a.e. on Γ_1 as an immediate consequence of Lemma 2.1. In other words, $\begin{pmatrix} \varphi_k \\ 0 \end{pmatrix} \in \mathcal{I}$.

(b) Let $(u^0, v^0) \in \mathcal{I}$. If $\partial_\nu u_b|_{\Sigma_1} = \partial_\nu u_0|_{\Sigma_1} = 0$, then $b\partial_t u_b|_{\Sigma_1} = 0$. Also, by the uniqueness of the solution of the IBVP (1), we conclude that $u_b = u_0$. Consequently,

$$(9) \quad b\partial_t u_0|_{\Sigma_1} = 0.$$

By our assumption the set where $\partial_t u_0|_{\Sigma_1}$ vanishes is of zero measure. Therefore, (9) implies that $b = 0$ a.e. on Γ_1 . \square

3. PROOF OF THEOREM 1.2

We begin by proving an extension lemma.

Lemma 3.1. (*Extension Lemma*) *Let k, ℓ two non negative integers. For any $g \in C^\ell([0, \tau]; H^{k+1/2}(\Gamma_1))$, there exists $G \in C^\ell([0, \tau]; H^{k+2}(\Omega))$ so that, for any $t \in [0, \tau]$,*

$$\begin{cases} \Delta G(t) = 0 & \text{in } Q, \\ G(t) = 0 & \text{on } \Gamma_0, \\ \partial_\nu G(t) = g(t) & \text{on } \Gamma_1. \end{cases}$$

and

$$(10) \quad \|G^{(j)}(t)\|_{H^{k+2}(\Omega)} \leq C \|g^{(j)}(t)\|_{H^{k+1/2}(\Gamma_1)}, \quad 0 \leq j \leq \ell.$$

Here the constant C is independent on g .

Proof. Let $h \in H^{k+1/2}(\Gamma_1)$. By Theorem 2.2 there exists a unique solution $Eh \in H^{k+2}(\Omega)$ of the BVP

$$\begin{cases} \Delta w = 0 & \text{in } Q, \\ w = 0 & \text{on } \Gamma_0, \\ \partial_\nu w = h & \text{on } \Gamma_1. \end{cases}$$

Moreover, we have the estimate

$$(11) \quad \|Eh\|_{H^{k+2}(\Omega)} \leq C \|h\|_{H^{k+1/2}(\Gamma_1)},$$

for some constant C independent on h .

If $g \in C^\ell([0, \tau]; H^{k+1/2}(\Gamma_1))$, then, using that $E : h \in H^{k+1/2}(\Gamma_1) \rightarrow Eh \in H^{k+2}(\Omega)$ is linear bounded operator (the fact that E is bounded is a consequence of estimate (11)), it is straightforward to check that $G(t) = Eg(t)$ satisfies the required properties. \square

Next, we consider the following non homogenous IBVP

$$(12) \quad \begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } Q, \\ u = 0 & \text{on } \Sigma_0, \\ \partial_\nu u = g & \text{on } \Sigma_1, \\ u(\cdot, 0) = u^0, \partial_t u(\cdot, 0) = v^0. \end{cases}$$

Proposition 1. *We assume that $g \in C^3([0, \tau]; H^{1/2}(\Gamma_1))$, $u^0 \in H^2(\Omega) \cap V$, $v^0 \in V$ and the compatibility condition*

$$(13) \quad \partial_\nu u^0 - g(\cdot, 0) = 0 \text{ on } \Gamma_1$$

holds. Then the IBVP (12) has unique solution u such that

$$\begin{pmatrix} u \\ u' \end{pmatrix} \in \mathcal{X} = C^1([0, \tau], H^1(\Omega) \times L^2(\Omega)) \cap C([0, \tau]; H^2(\Omega) \times H^1(\Omega))$$

and

$$(14) \quad \left\| \begin{pmatrix} u \\ u' \end{pmatrix} \right\|_{\mathcal{X}} \leq C \left(\left\| \begin{pmatrix} u^0 \\ v^0 \end{pmatrix} \right\|_{H^2(\Omega) \times H^1(\Omega)} + \|g\|_{C^3([0, \tau]; H^{1/2}(\Gamma_1))} \right).$$

Moreover, under the additional assumptions

$$(15) \quad g \in C^6([0, \tau]; H^{1/2}(\Gamma_1)), \quad \begin{pmatrix} u^0 - G(0) \\ v^0 - G'(0) \end{pmatrix} \in D(\mathcal{A}_0^4),$$

$u' \in C^3([0, \tau]; H^1(\Omega))$ and

$$(16) \quad \|u'\|_{C^3([0, \tau]; H^1(\Omega))} \leq C \left(\left\| \begin{pmatrix} u^1 \\ v^1 \end{pmatrix} \right\|_{D(\mathcal{A}_0^4)} + \|g\|_{C^6([0, \tau]; H^{1/2}(\Gamma_1))} \right).$$

Proof. We denote by $G \in C^3([0, \tau]; H^2(\Omega))$ the function given by Lemma 3.1 and corresponding to g . We observe that if u is the solution of the IBVP (12) then, $v = u - G$ is the solution of following one

$$(17) \quad \begin{cases} \partial_t^2 v - \Delta v = F & \text{in } Q, \\ v = 0 & \text{on } \Sigma_0, \\ \partial_\nu v = 0 & \text{on } \Sigma_1, \\ v(\cdot, 0) = u^1, \partial_t v(\cdot, 0) = v^1. \end{cases}$$

Here

$$F = G'', \quad u^1 = u^0 - G(0), \quad v^1 = v^0 - G'(0).$$

By the regularity assumptions on u^0 , v^0 and g and compatibility condition (13), we get that

$$F \in C^1([0, \tau]; L^2(\Omega)), \quad \begin{pmatrix} u^1 \\ v^1 \end{pmatrix} \in D(\mathcal{A}_0).$$

Therefore, the IBVP (17) has a unique solution v so that

$$\begin{pmatrix} v \\ v' \end{pmatrix} \in C^1([0, \tau], \mathcal{H}) \cap C([0, \tau]; D(\mathcal{A}_0)).$$

This solution is given by

$$(18) \quad \begin{pmatrix} v(t) \\ v'(t) \end{pmatrix} = e^{t\mathcal{A}_0} \begin{pmatrix} u^1 \\ v^1 \end{pmatrix} + \int_0^t e^{(t-s)\mathcal{A}_0} \begin{pmatrix} 0 \\ F(s) \end{pmatrix} ds.$$

In light of estimate (10), we have

$$(19) \quad \left\| \begin{pmatrix} v \\ v' \end{pmatrix} \right\|_{C^1([0,\tau];\mathcal{H}) \cap C([0,\tau];D(\mathcal{A}_0))} \leq C \left(\left\| \begin{pmatrix} u^0 \\ v^0 \end{pmatrix} \right\|_{H^2(\Omega) \times H^1(\Omega)} + \|g\|_{C^3([0,\tau];H^{1/2}(\Gamma_1))} \right).$$

Since $u = v + G$, we deduce that $\begin{pmatrix} u \\ u' \end{pmatrix} \in \mathcal{X}$ and (19) implies (14).

Next, we assume that the additional assumptions:

$$g \in C^6([0,\tau];H^{1/2}(\Gamma_1)), \quad \begin{pmatrix} u^1 \\ v^1 \end{pmatrix} \in D(\mathcal{A}_0^4),$$

hold. Then we deduce from (18) that $u' \in C^3([0,\tau];H^1(\Omega))$ and (16) is satisfied. \square

Proof of Theorem 1.2. We make the following assumption

$$\begin{pmatrix} u^0 \\ v^0 \end{pmatrix} \in \bigcap_{0 \leq \rho \leq 1} D(\mathcal{A}_{\rho b}^7).$$

According to regularity result (2), we have

$$u'_{\rho b}|_{\Gamma_1} \in C^6([0,\tau];H^{1/2}(\Gamma_1)), \quad 0 \leq \rho \leq 1.$$

and

$$(20) \quad \|u'_{\rho b}|_{\Gamma_1}\|_{C^6([0,\tau];H^{1/2}(\Gamma_1))} \leq C \left\| \begin{pmatrix} u^0 \\ v^0 \end{pmatrix} \right\|_{D(\mathcal{A}_0^7)}, \quad 0 \leq \rho \leq 1.$$

We see that $v_\rho = u_{\rho b} - u_0$ solves the IBVP (12) with $g = -\rho b u'_{\rho b}$. By using (16), we get

$$(21) \quad \|v'_\rho\|_{C^3([0,\tau];H^{1/2}(\Gamma_1))} = \|u'_{\rho b} - u'_0\|_{C^3([0,\tau];H^{1/2}(\Gamma_1))} \leq C\rho, \quad 0 \leq \rho \leq 1.$$

Next, let w_0 be the solution of the IBVP (12) corresponding to $u^0 = v^0 = 0$ and $g = -b u'_0$. Then $z = u_{\rho b} - u_0 - \rho w_0$ is the solution of the IBVP (12) corresponding to $u^0 = v^0 = 0$ and $g = -\rho b(u'_{\rho b} - u'_0)$. Hence

$$\|\partial_\nu z\|_{C^3([0,\tau];H^{1/2}(\Gamma_1))} = \rho \|b(u'_{\rho b} - u'_0)\|_{C^3([0,\tau];H^{1/2}(\Gamma_1))}.$$

This estimate, in combination with (21), yields

$$\|\partial_\nu z\|_{C^3([0,\tau];H^{1/2}(\Gamma_1))} \leq C\rho^2, \quad 0 \leq \rho \leq \rho_0.$$

Therefore,

$$\lim_{\rho \downarrow 0} \frac{\partial_\nu u_{\rho b} - \partial_\nu u_0}{\rho} = -b \partial_t u_0 \quad \text{in } C^3([0,\tau];H^{1/2}(\Gamma_1))$$

and then

$$\lim_{\rho \downarrow 0} \frac{\partial_\nu u_{\rho b} - \partial_\nu u_0}{\rho} = -b \partial_t u_0 \quad \text{in } L^2(\Sigma_1).$$

By using $2\kappa = \|b \partial_t u_0\|_{L^2(\Sigma_1)} \neq 0$, we get

$$\kappa \rho \leq \|\partial_\nu u_{\rho b} - \partial_\nu u_0\|_{L^2(\Sigma_1)}, \quad 0 \leq \rho \leq \rho_0,$$

for some $0 < \rho_0 \leq 1$.

We can rewrite this estimate as follows

$$\tilde{\kappa} \|\rho b - 0\|_{B_{1/2,1}(\Gamma_1)} \leq \|\partial_\nu u_{\rho b} - \partial_\nu u_0\|_{L^2(\Sigma_1)}, \quad 0 \leq \rho \leq \rho_0.$$

This completes the proof. \square

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